Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 4, pp. 713–724, 1993

0021–8928/93 $24.00+0.00

0021-8928(93)E0014-N

THE INTERACTION OF A VIBRATING PUNCH WITH A PRESTRESSED HALF-SPACE†

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(*Received* 6 *May* 1992)

A method of studying the dynamics of a rigid punch vibrating on the surface of a prestressed half-space is developed by generalizing [1, 2] the method of factorization for integral equations, with the symbols of the kernels having branch points on the real axis. The investigations are carried out within the framework of the linearized theory of the superposition of small deformations on to a finite deformation [3, 4]. A new approximate solution is constructed for the problem of the vibrations of a strip-like punch on the surface of a medium. The structure of this solution graphically exhibits the wave field under the punch and on the free surface and also enables one to carry out an efficient analysis of the effect of initial stresses on the wave process both under the punch and outside it. In constructing the solution, a special approximation [5] is used which takes account of all the characteristic features of the symbol of the kernels of the integral equation including branch points on the real axis. The vibration at the edges of the punch caused by the excitation of longitudinal and transverse waves by them in the prestressed medium, is demonstrated explicitly.

AN EFFICIENT method of studying the effect of initial stresses on the wave process in the contact zone and on the free surface was developed in [8–11] within the framework of the linearized theory of elastic waves [6, 7]. However, this method does not take account of the existence, in the kernel of the integral equation, of branch points on the real axis, which is typical of problems on the vibrations of a punch on the surface of a half-space. The effect of the branch points may be neglected [8–15] in the case of low-frequency vibrations of the punch, but these branch points must be taken into account at fairly high frequencies [1, 2, 5].

1. Three states (configurations) [3, 4, 6, 7] are distinguished when investigating processes in a prestressed body: the natural (unstressed) state, the initially deformed state and the perturbed state. Following [3, 4], let us introduce the system of coordinates x_1, x_2, x_3 associated with the initially-deformed state of the body which occupies the domain $x_3 \leqslant 0$, $-\infty \leqslant x_1$, $x_2 \leqslant \infty$.

The boundary-value problem on the excitation of a prestressed medium by an oscillating load $\mathbf{q}(x_1, x_2, t)$, distributed in the domain Ω is described by linearized equations of motion with boundary conditions [3, 4]

$$\begin{aligned} &\nabla \cdot \Theta = \rho \partial^2 \mathbf{u} / \partial t^2 \\ &\mathbf{n} \cdot \Theta = \mathbf{q}(x_1, x_2, t), \ x_3 = 0, \ x_1, x_2 \in \Omega \\ &\mathbf{n} \cdot \Theta = 0, \ x_3 = 0, \ x_1, x_2 \notin \Omega \end{aligned} \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 4, pp. 123–134, 1993.

$$\Theta = T\cdot\nabla\mathbf{u} + 4\sqrt{\frac{g}{G}}\left[-\psi_0\varepsilon(\mathbf{u}) + \psi_2 F\cdot\varepsilon(\mathbf{u})\cdot F + \sum_{k=0}^{2}\sum_{m=0}^{2} V_{km}F^kF^m\cdot\varepsilon(\mathbf{u})\right]$$

Here, ∇ is the Hamilton operator, defined in the coordinates of the initially deformed configuration, $\mathbf{u}$ is the displacement vector of an arbitrary point, ρ is the density of the medium, $\mathbf{n}$ is the outward normal to the surface of the medium, $\mathbf{q}$ is the specified stress vector, T is the initial stress tensor, G and g are the metric factors of the natural and initially deformed configurations respectively, F is the measure of the Finger deformation [3, 4] of the initial state and $\varepsilon(\mathbf{u})$ is the linear tensor of the perturbed state. The coefficients ψ_k participate in the representation of the equation of state of the material of the medium [3, 4]

$$T = 2\sqrt{\frac{g}{G}}(\psi_0 E + \psi_1 F + \psi_2 F^2) \tag{1.2}$$

and is defined in terms of the elastic potential. In the case of a hyperelastic, compressible medium, which has an elastic potential of the form ($I_k = I_k(F)$ are invariants of the measure of the initial deformation)

$$Э = Э(I_1, I_2, I_3) \tag{1.3}$$

the coefficients ψ_k and V_{ij} are defined by the formulae

$$\begin{aligned}&\psi_0 = I_3\frac{\partial Э}{\partial I_3},\quad \psi_1 = I_3\frac{\partial Э}{\partial I_1} + I_1\frac{\partial Э}{\partial I_2},\quad \psi_2 = -\frac{\partial Э}{\partial I_2}\\ &V_{0m} = I_3\frac{\partial \psi_m}{\partial I_3},\quad V_{1m}\frac{\partial \psi_m}{\partial I_1} + I_1\frac{\partial \psi_m}{\partial I_2},\quad V_{2m} = -\frac{\partial \psi_m}{\partial I_2},\quad m = 0,1,2\end{aligned} \tag{1.4}$$

Let us further assume that the oscillations are of a harmonic nature and all the functions have the form $f(x_1, x_2, x_3, t) = f'(x_1, x_2, x_3)e^{-i\omega t}$. We will subsequently omit primes and the exponential factor.

2. Let us assume that the initial deformation is homogeneous and defined by the relation [3, 4] ($\mathbf{R}$ and $\mathbf{r}$ are the radius vectors of a point of the medium in the initially deformed and natural configurations, respectively, $\upsilon_i = 1+\delta_i$, δ_i are the relative elongations of the fibres directed in the natural state along the axes x_i, $i = 1, 2, 3$, and δ_{ij} is the Kronecker delta)

$$\mathbf{R} = \mathbf{r}\cdot\Lambda,\quad F = \Lambda^T\cdot\Lambda,\quad \Lambda = [\delta_{ij}\upsilon_i]_{i,j=1,2,3}$$

In this case, the components of the tensor Θ have the form

$$\Theta_{ii} = 2(\det\Lambda)^{-1}\sum_{k=1}^{3} X_{ik}\frac{\partial u_k}{\partial x_k},\quad i = 1,2,3$$

$$\Theta_{ij} = 2(\det\Lambda)^{-1}\left[\upsilon_i^2\psi_{ij}\frac{\partial u_j}{\partial x_i} + \varphi_{ij}\frac{\partial u_i}{\partial x_j}\right],\quad i \neq j$$

$$2X_{ii} = a_i + b_{ii},\quad X_{ij} = b_{ij},\quad i \neq j$$

$$a_i = -\psi_0 + \psi_1\nabla_i^2 + 3\psi_2\nabla_i^4$$

$$b_{ij} = V_{00} + \upsilon_i^2\upsilon_j^2(V_{11} + V_{12}\upsilon_j^2 + V_{21}\upsilon_i^2)$$

$$\psi_{ij} = \psi_1 + \psi_2(\upsilon_i^2 + \upsilon_j^2)$$

$$\varphi_{ij} = -\psi_0 + \psi_2\upsilon_i^2\upsilon_j^2$$

Using the methods of the operational calculus and the limiting absorption principle, the solution of boundary-value problem (1.1) can be written in the form (α and β are the parameters of the Fourier transform with respect to the variables x_1 and x_2)

$$\mathbf{u}(x_1,x_2,x_3)=\frac{1}{4\pi^2}\iint_{\Omega}k(x_1-\xi,x_2-\eta,x_3,\omega)\mathbf{q}(\xi,\eta)d\xi d\eta \tag{2.1}$$

$$k(s,t,x_3,\omega)=\int_{\Gamma_1}\int_{\Gamma_2}K(\alpha,\beta,x_3,\omega)e^{i(\alpha s+\beta t)}d\alpha d\beta$$

The contours Γ_1 and Γ_2 are chosen in accordance with the limiting absorption principle [1, 2], and the behaviour of the elements of the matrix-function $K(\alpha,\beta,x_3,\omega)$ on the real axis, the properties thereof being determined by the nature of the initial deformed state and the properties of the material of the medium. In the case when the initial stressed state is specified by the condition (σ_{ii}^0 are the components of the tensor T (1.2)) $\sigma_{11}^0=\sigma_{22}^0\neq\sigma_{33}^0$ (state a), the matrix-function $K(\alpha,\beta,x_3,\omega)$ has the representation [1, 2, 12, 13] which is characteristic of dynamic problems in the theory of elasticity

$$K(\alpha,\beta,x_3,\omega)=\begin{Vmatrix}\alpha^2M+\beta^2N & \alpha\beta(M-N) & -i\alpha S\\ \alpha\beta(M-N) & \beta^2M+\alpha^2N & -i\beta S\\ i\alpha S & i\beta S & R\end{Vmatrix}$$

$$u=\sqrt{\alpha^2+\beta^2},\quad M=M(u,x_3,\omega),\quad N=N(u,x_3,\omega),\quad S=S(u,x_3,\omega),\quad R=R(u,x_3,\omega)$$

$$M=\left[\sigma_1m_2e^{\sigma_1x_3}-\sigma_2m_1e^{\sigma_2x_3}\right]u^{-2}\Delta^{-1},\quad N=e^{\sigma_1x_3}u^{-2}l_3^{-1}$$

$$S=[s_1-s_2]\Delta^{-1},\quad R=[l_1t_2-l_2t_1]\Delta^{-1},\quad \Delta=l_1m_2-m_1l_2$$

$$t_k(u,x_3)=d_ke^{\sigma_kx_k},\quad s_k(u,x_3)=l_k\sigma_{3-k}e^{\sigma_kx_k},\quad k=1,2$$

Here

$$\begin{gathered}l_k=A_3\sigma_k^2-A_8d_k,\quad m_k=[A_6d_k+A_4u^2]\sigma_k\\ d_k=(A_3\sigma_k^2-R_1)/A_5,\quad k=1,2,\quad l_3=A_3\sigma_3^2\\ \sigma_k^2=(D_2\pm\sqrt{\Sigma})/2D_1,\quad k=1,2,\quad \sigma_3^2=R_2/A_3,\quad D_1=A_3A_6\\ D_2=A_3R_3+A_6R_1-A_5^2u^2,\quad D_3=R_1R_3,\quad \Sigma=D_2^2-4D_1D_3\\ R_k=A_ku^2-\rho\omega^2,\quad k=1,2,\quad R_3=A_7u^2-\rho\omega^2\\ A_1=g_0\mathrm{X}_{11},\quad A_2=g_0\upsilon_2^2\psi_{12},\quad A_3=g_0\upsilon_3^2\psi_{13}\\ A_4=2g_0\mathrm{X}_{13},\quad A_5=g_0(\varphi_{13}+2\mathrm{X}_{13}),\quad A_6=g_0\mathrm{X}_{33}\\ A_7=g_0\upsilon_1^2\psi_{13},\quad A_8=g_0\varphi_{13},\quad g_0=2(\det\Lambda)^{-1}\end{gathered}\tag{2.2}$$

In the case when $\sigma_{11}^0\neq\sigma_{22}^0=\sigma_{33}^0$ (state b), the elements of the matrix function $K(\alpha,\beta,x_3,\omega)=\|K_{ij}\|_{i,j=1,2,3}$ have the form

$$K_{11}=\{d_1f_{11}^0-d_2f_{12}^0\}\Delta^{-1},\quad K_{12}=\alpha\beta\{f_{11}^0-f_{12}^0+\sigma_3\beta^{-2}f_{13}^0\}\Delta^{-1}$$

$$K_{13}=-i\alpha\{\sigma_1f_{11}^0-\sigma_2f_{12}^0+f_{13}^0\}\Delta^{-1},\quad K_{21}=K_{12}$$

$$K_{22}=\{f_{21}^0-f_{22}^0+\sigma_3\beta^{-2}f_{23}^0\}\beta^2\Delta^{-1},\quad K_{23}=-i\beta\{\sigma_1f_{21}^0-\sigma_2f_{22}^0+f_{23}^0\}\Delta^{-1}$$

$$K_{31}=-K_{13},\ K_{32}=-K_{23},\ K_{33}=\{\sigma_1f_{31}^0-\sigma_2f_{32}^0+f_{33}^0\}\Delta^{-1}$$

$$\Delta=\sigma_3l_3f_{33}-l_2f_{32}+l_1f_{31}$$

$$f_{1k}=\sigma_{3-k}\sigma_3n_{3-k}l_3-l_{3-k}n_3,\quad f_{2k}=l_{3-k}m_3-\sigma_{3-k}\sigma_3m_{3-k}l_3$$

$$
\begin{aligned}
&f_{3k} = \sigma_{3-k}[m_{3-k}n_3 - n_{3-k}m_3], \quad k=1,2\\
&f_{13} = n_1[\sigma_1 l_2 - l_1\sigma_2], \quad f_{23} = \sigma_2 m_2 l_1 - l_2\sigma_1 m_1\\
&f_{33} = \sigma_1\sigma_2 n_1[m_1 - m_2], \quad f_{ik}^0 = f_{ik}e^{\sigma_k x_3}, \quad i,k = 1,2,3\\
&m_k = A_2 k_k + \alpha^2 A_8, \quad l_k = A_6\sigma_k^2 - (A_4 d_k + \beta^2 A_{42})\\
&n_k = \beta^2(A_{82} + A_{32}), \quad d_k = (A_6\sigma_k^2 - R_6)/A_5, \quad k=1,2\\
&m_3 = \alpha^2 A_8, \quad l_3 = A_6 - A_{42}, \quad n_3 = A_{32}\sigma_3^2 + \beta^2 A_8\\
&\sigma_{1,2}^2 = [D_2 \pm \Sigma^{1/2}][2D_1]^{-1}, \quad \sigma_3^2 = R_4/A_{32}\\
&D_1 = A_2 A_6, \quad D_2 = A_{32}R_5 + A_2 R_4 + A_{52}A_2\beta^2 + R_7\\
&D_3 = R_4 R_5 + R_7\beta^2, \quad \Sigma = D_2^2 - 4D_1 D_3,\\
&R_4 = A_7\alpha^2 + A_{32}\beta^2 - \rho\omega^2\\
&R_5 = A_1\alpha^2 + A_2\beta^2 - \rho\omega^2,\\
&R_6 = A_7\alpha^2 + A_6\beta^2 - \rho\omega^2\\
&R_7 = (A_{52}A_1 - A_5^2)\alpha^2 + A_{52}A_2\beta^2 - \rho\omega^2\\
&A_{32} = g_0 \upsilon_3^2 \psi_{23}, \quad A_{52} = g_0(\varphi_{23} + 2X_{23})\\
&A_{82} = g_0\varphi_{23}, \quad A_{42} = 2g_0 X_{23}
\end{aligned}
\tag{2.3}
$$

($A_1, \ldots, A_8$ are defined in the last formulae of (2.2)).

In the general case when $\sigma_{11}^0 \neq \sigma_{22}^0 \neq \sigma_{33}^0 (\upsilon_1 \neq \upsilon_2 \neq \upsilon_3)$, the elements of the matrix-function $K(\alpha, \beta, x_3, \omega)$ are significantly more cumbersome.

3. Expression (2.1) defines the displacement of an arbitrary point of the medium acted on by a specified load $\mathbf{q}(x_1, x_2)$. In the case of the problem of the vibration of a punch on the surface of a prestressed half-space, it is necessary to put $x_3 = 0$ in expressions (2.1). Here, $\mathbf{q}(x_1, x_2)$ is an unknown function of the contact stress distribution and $\mathbf{u}(x_1, x_2) = \mathbf{f}(x_1, x_2)$ is the specified amplitude of the displacements of the punch base. The problem of the vertical vibrations of a strip-like punch, which, in the plane view, occupies a domain $|x_1| \leq a$ on the surface of the prestressed half-space (the case of shear vibrations has been investigated in [16]) is a special case. Its solution reduces to a system of integral equations ($\mathbf{f}(x) = \{f_1(x), f_3(x)\}$, $\mathbf{q}(x) = \{q_1(x), q_3(x)\}$ are the displacement and stress vectors, respectively) which can be written in the dimensionless form

$$
\mathbf{f}'(x_1') = \frac{1}{2\pi}\int_{-1}^{1} k(x_1' - \xi')\mathbf{q}'(\xi')d\xi', \quad |x_1'| \leq 1 \tag{3.1}
$$

$$
k(t') = \int_\Gamma K(\alpha)e^{i\alpha t}d\alpha, \quad K(\alpha) = \|K_{ij}(\alpha)\|_{i,j=1,3}
$$

$$
\left(x_i' = \frac{x_i}{a}, \quad \xi_i' = \frac{\xi_i}{a}, \quad t_i' = \frac{t_i}{a}, \quad q_i' = \frac{q_i}{\mu}, \quad f_i' = \frac{f_i}{a}\right)
$$

We will henceforth omit the primes. In the case of an initial state a

$$
K_{11}(\alpha) = M^0(\alpha), \quad K_{13}(\alpha) = -i\alpha S^0(\alpha), \quad K_{31}(\alpha) = -K_{13}(\alpha), \quad K_{33}(\alpha) = R^0(\alpha) \tag{3.2}
$$

where

$$M^0(\alpha)=(\sigma_1 m_2-\sigma_2 m_1)\Delta^{-1} \tag{3.3}$$

$$S^0(\alpha)=(l_1\sigma_2-l_2\sigma_1)\Delta^{-1},\quad R^0(\alpha)=(l_1 d_2-l_2 d_1)\Delta^{-1}$$

The coefficients l_k, m_k, d_k, σ_k are defined by formulae (2.2).

In the case of an initial state b

$$K_{11}(\alpha)=R^0(\alpha),\quad K_{13}(\alpha)=i\alpha S^0(\alpha),\quad K_{31}(\alpha)=-K_{13}(\alpha),\quad K_{33}(\alpha)=M^0(\alpha) \tag{3.4}$$

The functions M^0, S^0 and R^0 are defined by formulae (3.3), but the coefficients l_k, m_k, d_k and σ_k are defined by formulae (2.3).

Expression (3.1) is a system of first-order integral equations in the unknown functions of the contact stresses $\mathbf{q}(x_1, x_2)$. The functions M^0, S^0, T^0 and R^0 (3.3) are analytic in the complex plane with non-intersecting cuts which strictly lie in the first and third quadrants (there are two in each of them) and have two poles on the real axis (one each on the positive and negative semi-axes). The cuts go from the branch points, which are determined numerically from the equation $\sigma_k(\alpha, \beta, \omega, \sigma_{11}^0, \sigma_{22}^0, \sigma_{33}^0)=0$ $(k=1, 2, 3)$, to infinitely remote points. These functions exhibit qualitatively different behaviour on the different segments of the real axis: when $|\alpha|\geqslant\kappa_2(\kappa_2>\kappa_1)$ they take real values, when $|\alpha|\leqslant\kappa_1$, they are purely imaginary and, when $\kappa_1<|\alpha|<\kappa_2$, they are complex. The representation $|\alpha|f(\alpha)=c_i+O(\alpha^{-2})$ holds for the functions R^0 and M^0 and for the function $S^0(\alpha)-\alpha^2S^0(\alpha)=c_3+O(\alpha^{-2})$ when $\alpha\to\infty$. Here c_i $(i=1, 2, 3)$ are functions of the initial stresses.

4. A detailed investigation of the effect of the initial deformation on the interaction between a punch and a prestressed medium can be carried out exclusively by analysing the solutions of the integral equation of the corresponding problems. As an example, let us consider the problem of the vertical vibrations of a punch, which is strip-like in plan view, on the surface of a half-space. There is no friction between the punch and the half-space. Let us suppose that the initial stress–strain state of the half-space is uniform and is determined by the condition $\sigma_{11}^0\neq\sigma_{22}^0\neq\sigma_{33}^0$.

The problem reduces to solving the integral equation

$$u_3(x_1)=\frac{1}{2\pi}\int_{-1}^{1}k_{33}(x_1-\xi)q(\xi)d\xi=f(x),\quad |x_1|\leqslant 1 \tag{4.1}$$

$$k_{33}(t)=\int_\Gamma K_{33}(\alpha)e^{i\alpha t}d\alpha \tag{4.2}$$

the function $K_{33}(\alpha)$ is defined by formulae (3.3) and (3.4) with the coefficients (2.4). The contour Γ coincides with the real axis, only deviating from it when going downwards around the positive singularities of $K_{33}(\alpha)$, the properties of which have been noted above, and upwards around the negative singularities.

By virtue of the properties of $K_{33}(\alpha)$ described above, a number of numerical and asymptotic methods ([5, 8–16], for example) may be used to solve the integral equation (4.1). Following the approach proposed in [1, 2], we replace $K_{33}(\alpha)$ by the function $K^*(\alpha)$ [5]

$$K^*(\alpha)=\frac{A\sqrt{\alpha^2-\kappa_1^2}}{(\alpha^2-\xi^2)(\alpha^2-a_0^2)(\alpha^2-\bar{a}_0^2)}M_+(\alpha)M_-(\alpha),\quad \operatorname{Im}a_0>0$$

$$M_\pm(\alpha)=B\alpha\sqrt{\alpha\pm\kappa_1}\sqrt{\alpha\pm\kappa_2}+(\sqrt{2}\alpha\pm\kappa_2)^2 \tag{4.3}$$

The function $K^*(\alpha)$ retains all the properties of $K_{33}(\alpha)$ and exhibits the same qualitative behaviour on the real axis. The poles $\alpha=\pm a_0$ and $\alpha=\pm\bar{a}_0$ compensate for the zeros of the functions $M_+(\alpha)$ and $M_-(\alpha)$. The constants A, B $(A>0,\ B>0)$ are chosen from the condition that the functions $K^*(\alpha)$ and $K_{33}(\alpha)$ should be equal to each other at zero and infinity.

It is clear that the function $K^*(\alpha)$ admits of the factorization

$$K^*(\alpha) = K_+(\alpha)K_-(\alpha) \tag{4.4}$$

$$K_\pm(\alpha) = \frac{d^\pm\sqrt{\alpha \pm \kappa_1}}{(\alpha \pm \xi)(\alpha \mp a_0)(\alpha \mp \bar{a}_0)} M_\pm(\alpha), \quad d^\pm = \sqrt{Ae^{\pm i\pi/4}}$$

$d^\pm$ are chosen in such a way that the condition $K_+(\alpha) = K_-(\alpha)$ is satisfied.

Taking account of the properties of the function $K^*(\alpha)$ noted above, we reduce Eq. (4.1), after some algebra [12, 13], to a system of second-order integral equations of the following form (the upper and then the lower signs are taken in succession)

$$X(z,\pm) = -\frac{1}{2\pi i}\int_{\Gamma_1} X(\alpha,\pm)P(\alpha,z)d\alpha + \alpha(z,\pm), \quad \operatorname{Im} z \leqslant 0 \tag{4.5}$$

$$P(\alpha,z) = K_-(\alpha)e^{-2i\alpha}[K_+(\alpha)(\alpha+z)]^{-1}$$

$$\alpha(z,\pm) = -\frac{1}{2\pi i}\int_{\Gamma_1}[F(\alpha) \pm F(-\alpha)][K_+(\alpha)(\alpha+z)]^{-1}e^{i\alpha}d\alpha$$

$$X(\alpha,\pm) = [\Phi^+(\alpha) \pm \Phi^-(\alpha)]e^{i\alpha}[K_-(\alpha)]^{-1}$$

in the auxiliary unknowns $X(z,\pm)$ which are combinations of $\Phi^+(\alpha)$, $\Phi^-(\alpha)$, i.e. of the Fourier transforms of functions which are continuations of the right-hand side of Eq. (4.1) in the domains $x > 1$ ($\varphi^+(x)$) and $x < 1$ ($\varphi^-(x)$), respectively and $F(\alpha)$ is the Fourier transform of the function $f(x)$. The contour Γ_1 lies strictly above the contour Γ, but does not leave the regularity zone which is a certain neighbourhood of the contour Γ [1, 2]. The solution of the integral equation (4.1) is determined by the formula [12, 13]

$$q(x) = \frac{1}{2\pi}\int_{\Gamma}[F(\alpha) + \Phi^+(\alpha) + \Phi^-(\alpha)]e^{-i\alpha x}[K(\alpha)^{-1}d\alpha, \quad |x| \leqslant 1 \tag{4.6}$$

$$\Phi^\pm(\alpha) = \frac{1}{2}[X(\mp\alpha,+) \pm X(\mp\alpha,-)]K_\pm(\alpha)e^{\pm i\alpha}$$

The behaviour of the free surface outside the punch has the form

$$\varphi^\pm(x) = \frac{1}{2\pi}\int_{\Gamma}\Phi^\pm(\alpha)e^{-i\alpha x}d\alpha, \quad |x| > 1 \tag{4.7}$$

In order to construct a solution of (4.5) we deform the contour Γ_1 in the lower half-plane (in the domain of regularity of $X(\alpha,\pm)$ $K_-(\alpha)$) in order that it passes around the cuts from the branch points $-\kappa_1$, $-\kappa_2$ to infinitely remote points parallel to the imaginary axis from $-\kappa_1 - i\infty$ to $-\kappa_1$ and from $-\kappa_2 - i\infty$ to $-\kappa_2$ on the left of the cuts and from $-\kappa_1$ to $-\kappa_1 - i\infty$ and from $-\kappa_2$ to $-\kappa_2 - i\infty$ on the right of the cuts). On representing the integrals over the left and right edges of the cuts in the form of a sum and taking account of the relation between the values of $K_+(\alpha)$ on these edges, we represent system (4.5) in the form [16]

$$X(z,\pm) = \pm\sum_{k=1}^{N+M} X(-z_k,\pm)\frac{r_k}{z - z_k} + \alpha(z,\pm) + O(e^{-2iz_{N+M+1}}), \quad \operatorname{Im} z \leqslant 0 \tag{4.8}$$

$$r_k = \frac{1}{\pi i}\frac{K_-(-z_k)}{K_+(-z_k)}e^{2iz_k}\Delta z_k$$

where $z_k = \kappa_1 + it_k$ $(k = 1, 2, \ldots, N)$ are mesh points along the edges of the cut, $[-\kappa_1, -\kappa_1 - i\infty]$ and $z_k = \kappa_2 + it_k$ $(k = N+1, \ldots, N+M)$, are mesh points along the edges of the cut $[-\kappa_2, -\kappa_2 - i\infty]$. Letting $z = -z_n$ $(n = 1, 2, \ldots, N+M)$, we obtain a finite system of algebraic equations in $X(z_k, \pm)$ [16], the solution of which, when substituted into (4.8), yields the formula

$$X(z,\pm) = \pm \sum_{k=1}^{N+M} \frac{r_k}{z - z_k} \left[\sum_{l=1}^{N+M} S_{kl}^{\pm} \alpha(-z_l, \pm) \right] + \alpha(z,\pm), \quad \operatorname{Im} z \leqslant 0 \tag{4.9}$$

where $S_{kl}^{\pm}$ are elements of the matrix which is the inverse of the matrix of the form

$$A^{\pm} = \left(\delta_{lk} \pm \frac{r_k}{z_l + z_k} \right)_{k,l=1,\ldots,N+M}$$

The functions $q(x)$ and $\varphi(x)$, $\psi(x)$ are recovered using formulae (4.6), (4.7).

5. Let us consider the case when the displacement of the base of the punch is specified by the function

$$f(x) = e^{i\eta x}, \quad |x| \leqslant 1 \tag{5.1}$$

Following [1, 2, 12, 13], we continue this function to the intervals $x > 1$ and $x < -1$. We thereby change to the new functions $\varphi^*(x)$ and $\psi^*(x)$

$$\varphi^*(x) = \varphi(x) - e^{i\eta x}, \quad x > 1, \quad \psi^*(x) = \psi(x) - e^{i\eta x}, \quad x < -1 \tag{5.2}$$

Then the functions $\alpha(z, \pm)$ on the right-hand sides of Eqs (4.8) are represented in the form

$$\alpha(z,\pm) = -\frac{1}{2} \frac{e^{\pm i\eta}}{K_{\mp}(\eta)(z \mp \eta)} \tag{5.3}$$

Taking account of relationships (4.5) and (4.8), and applying formulae of the operational calculus [17, 18] to (4.6) and (4.7), we obtain

$$q(x) = -\frac{e^{i\eta x}}{k(\eta)} + q^+(x) + q^-(x) \tag{5.4}$$

$$q^{\pm}(x) = \sum_{n=1}^{2} \left[\sum_{m=0}^{3} q_{mn}^{\pm}(x) + \sum_{k=1}^{N+M} P_{kn}^{\pm}(x) \right] \tag{5.5}$$

$$q_{0n}^{\pm}(x) = G_{0n}^{\pm} e^{i\kappa_n(1 \mp x)} [\pi(1 \mp x)]^{-1/2} \tag{5.6}$$

$$q_{mn}^{\pm}(x) = G_{mn}^{\pm} \varphi_n^{\pm}(t_m^{\pm}), \quad m = 1,2,3$$

$$P_{kn}^{\pm}(x) = \frac{1}{2} b_k^{\pm} K_{+}^{-1}(z_k) e^{-iz_k(1 \mp x)} + B_{kn}^{\pm} \varphi_n^{\pm}(z_k) \tag{5.7}$$

Here

$$\varphi_n^{\pm}(z) = e^{-iz(1 \mp x)} \operatorname{erf} \sqrt{-i(\kappa_n + z)(1 \mp x)} [-i(\kappa_n + z)]^{-1/2}$$
$$G_{mn}^{\pm} = \gamma_{1n} \delta_m \left[\alpha_{m\eta}^{n\pm} \frac{e^{\pm i\eta}}{K_{\mp}(\eta)} - \frac{1}{2} \sum_{k=1}^{N+M} b_k^{\pm} \alpha_{mk}^{n} \right], \quad m = 0,1,2,3, \quad n = 1,2 \tag{5.8}$$

$$B_{kn}^{\pm} = -\frac{i}{1}\gamma_{1n}\alpha_{1k}^{n}b_k^{\pm}, \quad G_{1n}^{\pm} = \gamma_{1n}\alpha_{1\eta}^{n\pm}\frac{e^{\pm i\eta x}}{K_{\pm}\ (\eta)}$$

$$b_k^{\pm} = r_k\sum_{l=1}^{N+M}[S_{kl}^{+}\alpha(-z_l,+)\mp S_{kl}^{-}\alpha(-z_l,-)]$$

$$\gamma_{1n} = d_n e^{-i\pi/4}, \quad \delta_0 = i, \quad \delta_n = 1, \quad n = 1,2,3$$

$$d_1 = \frac{B}{d^{+}(B^2-4)}, \quad d_2 = \frac{1}{d^{+}(B^2-4)}, \quad t_1^{\pm} = \pm\eta, \quad t_2^{\pm} = -b_0, \quad t_3^{\pm} = -\bar{b}_0$$

The behaviour of the free surface outside the punch is described by the function

$$\varphi(x) = \varphi^{\pm}(x), \quad 1\mp x<0 \tag{5.9}$$

$$\varphi^{\pm}(x) = \sum_{n=1}^{2} D_{1n}^{\pm}\Psi_n^{\pm}(\pm\eta) + D_{2n}^{\pm}\Psi_n^{\pm}(\xi) + D_{3n}^{\pm}\Psi_n^{\pm}(a_0) + D_{4n}^{\pm}\Psi_n^{\pm}(\bar{a}_0) + $$
$$+\sum_{k=1}^{N+M} C_{kn}^{\pm}\Psi_n^{\pm}(z_k) \tag{5.10}$$

$$\Psi_n^{\pm}(z) = e^{-iz(1\mp x)}\operatorname{erf}\sqrt{-i(\kappa_n z)(-1\pm x)}[-i(\kappa_n - z)]^{-1/2} \tag{5.11}$$

$$D_{mn}^{\pm} = \gamma_{2n}e^{-i\pi/4}\left[\beta_{m\eta}^{n\pm}\frac{e^{\pm i\eta}}{K_{\mp}(\eta)} - \frac{i}{2}\sum_{k=1}^{N+M} b_n^{\pm}\beta_{mk}^{n}\right], \quad m = 2,3 \tag{5.12}$$

$$D_{1n}^{\pm} = \gamma_{2n}e^{-i\pi/4}\beta_{1n}^{n\pm}, \quad \gamma_{2n} = Bd^{-}\sqrt{A}, \quad C_{kn}^{\pm} = -\frac{1}{2}\gamma_{1n}\beta_{1k}^{n}b_k^{\pm}$$

The coefficients $\alpha_{m\eta}^{n\pm}$, α_{mk} are determined from an expansion in simple fractions of the expressions

$$[(\alpha\pm\eta)K_{+}(\alpha)]^{-1} \quad \text{and} \quad [(\alpha - z_k)K_{+}(\alpha)]^{-1}$$

respectively, while the coefficients $\beta_{m\eta}^{n\pm}$ and β_{mk} are determined from the expansion

$$K_{-}(\alpha)[\alpha\pm\eta]^{-1} \quad \text{and} \quad K_{-}(\alpha)[\alpha\pm z_k]^{-1}$$

respectively. The functions $K_{\pm}(\alpha)$ are defined by formulae (4.4). The constants a_0, $\bar{a}_0$, b_0, $\bar{b}_0$ which occur in (5.7) and (5.10) are zeros of $M_{+}(\alpha)$ (4.3) and

$$L(\alpha) = B\alpha\sqrt{\alpha+\kappa_1}\sqrt{\alpha+\kappa_2} - (\sqrt{2\alpha}+\kappa_2)^2$$

respectively.

It is seen from the representation (5.6) that the initial stresses have an effect on the singularities of an oscillatory character.

It follows from (5.5) and (5.7) that rapidly decaying waves move from the edges of the punch with velocities equal to those of the longitudinal and transverse waves in the prestressed medium. Similarly, rapidly decaying waves which move from its edges and have the velocities of longitudinal and transverse waves are developed on the free surface outside the punch (5.9). Apart from these waves, one non-decaying wave (Im $\xi = 0$) propagates in each of the two directions along the surface from the edges of the punch, with the initial stresses having an effect on the phase velocity of this wave as well as on the velocities of the rapidly decaying waves.

6. As an example, let us consider the problem of the vibration of a plane punch ($\eta=0$). For this purpose, it is necessary to put $\eta=0$ in formulae (5.1), (5.2), (5.4)–(5.8) and (5.9)–(5.12). Here, there is no appreciable simplification of expressions (5.4)–(5.12). The validity of the equalities

$$D_{mn}^{+}=D_{mn}^{-}=D_{mn},\quad C_{kn}^{+}=C_{kn}^{-}=C_{kn},\quad G_{mn}^{+}=G_{mn}^{-}=G_{mn}$$
$$b_k^0=b_k^0=b_k,\quad B_{kn}^{+}=B_{kn}^{-}=B_{kn}$$

follows from (5.3), (5.8) and (5.12) which determines the symmetry of the wave process on the surface of the medium. We note that, when $\eta=0$ in the unstressed case, the degenerate component of the function $q(x)$ (5.4) is qualitatively the same as the asymptotic solution [5] in the case of high-frequency vibrations.

The amplitude of the reactive force, acting on the planar punch ($\eta=0$) from the half-space, has the form

$$P=\int_{-1}^{1}q(x)dx=P_1+P_2,\quad P_n=P_{00n}+P_{0n}+P_{1n} \tag{6.1}$$

where

$$P_{00n}=-\frac{2}{K(0)}+\sum_{k=1}^{N+M}\frac{b_k}{K_+(z_k)}\frac{1-e^{-2ik_k}}{z_k}+2iG_{1n}\frac{\sqrt{2}}{\kappa_n\sqrt{\pi}}e^{2i\kappa_n}$$
$$P_{0n}=2\Phi_n(0)\left[G_{0n}+2\left(1+\frac{1}{4i\kappa_n}\right)G_{1n}+i\sum_{m=1}^{3}\frac{G_{mn}}{t^+m}-\frac{i}{2}\sum_{k=1}^{N+M}\frac{B_{km}}{z_k}\right] \tag{6.2}$$

$$P_{1n}=-2i\sum_{m=2}^{3}\frac{G_{mn}}{t^+m}\Phi_n(t_m^+)+i\sum_{k=1}^{N+M}\frac{B_{km}}{z_k}\Phi_n(z_k)$$
$$\Phi_n(x)=e^{-ix}\operatorname{erf}\sqrt{-2i(\kappa_n+x)}\,[-i(\kappa_n+x)]^{-1/2},\quad n=1,2 \tag{6.3}$$

Formulae (5.4)–(5.8) and (5.9)–(5.12) quite graphically represent the structure of the wave field both under the punch and on the free surface. The parameters κ_k ($k=1$, 2) are related to the initial deformation in a complex way, and the initial stresses therefore have an immediate effect on the nature of the wave field.

7. Formulae (5.4)–(5.12) and (6.1)–(6.3) are constructed in the most general case, regardless of the form of the initial stress state and the properties of the material of the medium. In order to carry out further investigations it is now necessary to specify the properties of the medium. For this purpose, we assume that the material of the medium is compressible, initially isotropic and has the elastic potential (1.3). As the latter, we use the Murnaghan potential [3, 4]

$$Э=\frac{1}{4}\left[\left(-3\lambda-2\mu+\frac{9}{2}l+\frac{n}{2}\right)I_1(F)+\frac{1}{2}(\lambda+2\mu-3l-2m)I_1(F)+\left(-2\mu+3m-\frac{n}{2}\right)I_2(F)-\right.$$
$$\left.-mI_1(F)I_2(F)+\frac{1}{6}(l+2m)I_1^3(F)+\frac{n}{2}(I_3(F)-1)\right]$$

(λ and μ are Lamé constants, and l, m and n are third-order constants). Assuming that the initial deformation is uniform ($\upsilon_i=\text{const}$), we obtain from (1.4)

$$\psi_0=\frac{1}{8}nJ_3$$

$$\psi_1=\frac{1}{4}\left[(J_1-3)\lambda-2\mu+\frac{1}{2}l(J_1-3)^2+m(J_1-J_2)-\frac{1}{2}n(J_1-1)\right]$$

$$\psi_2 = \frac{1}{4}\left[2\mu + m(I_1 - 3) + \frac{1}{2}n\right]$$

A material with the following parameters was used for the numerical calculations: $\rho = 7.748\times10^3$ kg/m^2, $\lambda = 1.1\times10^{11}$ N/m^2, $\mu = 0.804\times10^{11}$ N/m^2, $l = -6.32\times10^{11}$ N/m^2, $m = -3.25\times10^{11}$ N/m^2, $n = -8.04\times10^{11}$ N/m^2 (09G2S steel [3, 4]).

The initial stressed state was assumed to be uniaxial and defined by the condition $\sigma_{11}^0 = p$, $\sigma_{22}^0 = \sigma_{33}^0 = 0$.

Graphs of the functions ($q^0(x) = q(x)$ without any initial deformation of the medium) Re $q^0(x)$ and Im $q^0(x)$ are shown in Figs 1 and 2 for various values of the vibrational frequency of the punch (curves 1–5 correspond to the values $\kappa_2 = 0.9$, 3.0, 4.5, 7.5 and 10.5). It is seen that, at the low frequency $\kappa_2 = 0.9$, the distribution of the contact stresses is close to the static one (the function Re $q^0(x)$ has a constant sign and is significantly greater than Im $q^0(x)$). At the intermediate frequency ($\kappa_2 = 3.0$), Im $q^0(x)$ increases sharply and Re $q^0(x)$ changes sign in the contact region while retaining its monotonic form. At high frequencies (curves 3–5) the plot of the contact stresses acquires an oscillatory form due to the fact that the wavelength of the shear wave excited by the edges of the punch becomes less than its size. The effect of the longitudinal wave occurs in the curve of the stress graph (curves 4, 5). The superposition of the oscillating terms on the penetrating (constant at fixed frequency) component $q^0(x)$ transforms Im $q^0(x)$ initially to a saddle-shaped form and, then, as the frequency is increased further, to a serrated form.

The functions Re $\eta(x)$ (Fig. 3) and Im $\eta(x)$ (Fig. 4), where $\eta(x) = [q^0(x) - q(x)]1\times10^3$ is the change in the contact stresses, are shown as a function of frequency (curves 1–5 correspond to the same values of κ_2 as in Figs 1 and 2) at a fixed initial deformation. At a low frequency, the effect of the initial deformation has a monotonic form over the entire domain of contact with the exception of the edges of the punch. the effect of the initial deformation becomes more complex at intermediate frequencies. For instance, already when $\kappa_2 = 3.0$ (curve 2), Re $\eta(x)$ and Im $\eta(x)$ change sign in the contact region. At high frequencies (curves 4 and 5), the effect of the initial deformation acquires an oscillatory character and the amplitude of the oscillation increases with frequency.

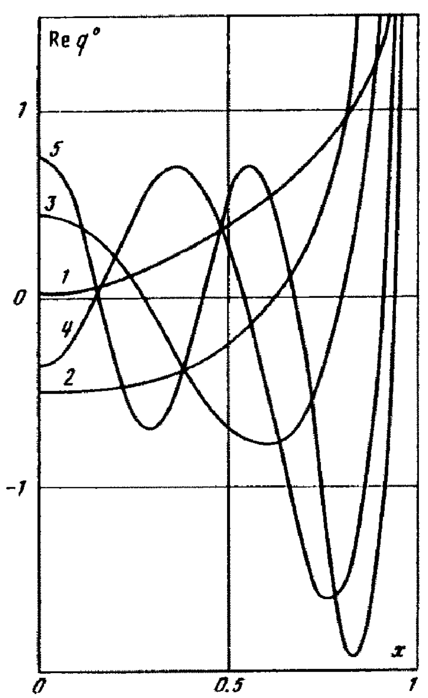

FIG. 1.

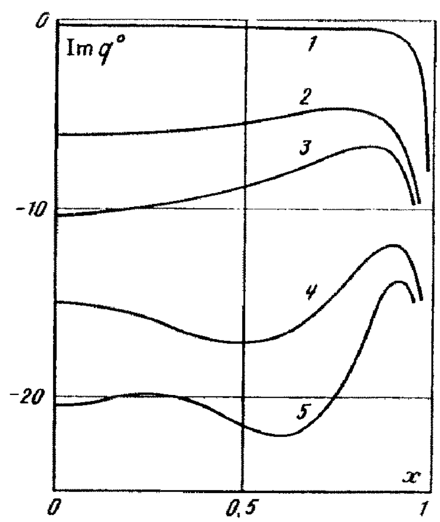

FIG. 2.

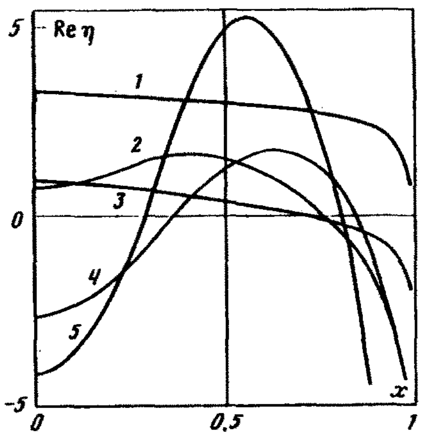

Fig. 3.

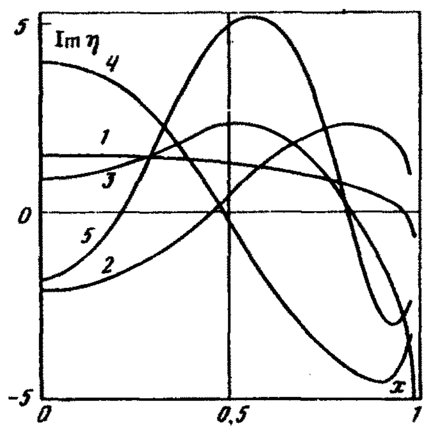

Fig. 4.

The functions Re $\eta(x)$ (the solid curves) and Im $\eta(x)$ (the dashed curves) are shown in Figs 5 and 6 as a function of the magnitude of the initial stresses (curves 1, 2, and 3 correspond to the values $p=2.5\times10^{-4}$, 5×10^{-4} and 10^{-3}) at fixed values of the frequency: $\kappa_2=4.5$ (Fig. 5), $\kappa_2=7.5$ (Fig. 6). It is seen that there are points under the punch where the initial deformation has no effect, as well as points where the effect of the initial deformation is a maximum. The number and locations of these points depend very much on the frequency. As the frequency increases, the number of points where the initial deformation has no effect on Re $q(x)$ and Im $q(x)$ increases.

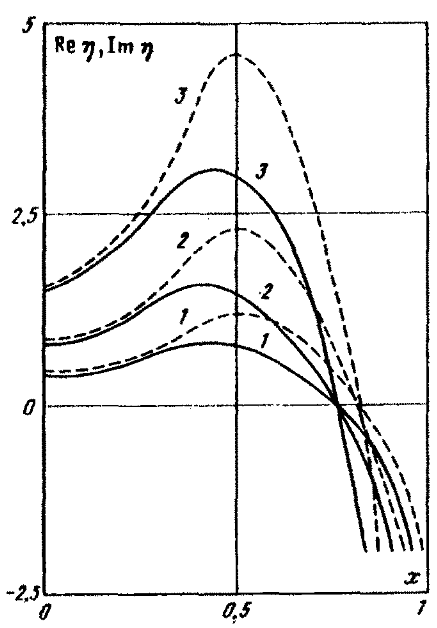

Fig. 5.

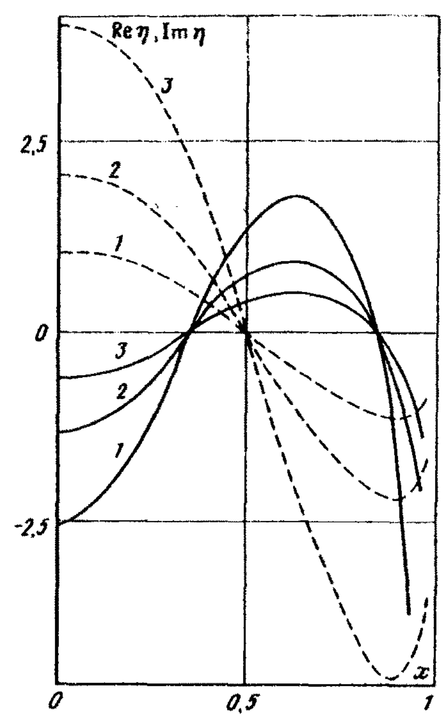

Fig. 6.

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Translated by E.L.S.